

Mathematics

On the Almost Everywhere Convergence and Partial Sum's Majorant of Series with Respect to Block-Orthonormal Systems

Givi Nadibaidze

I. Javakhishvili Tbilisi State University

(Presented by Academy Member E. Nadaraya)

ABSTRACT. In the present paper the behaviour of partial sum's majorant of series with respect to block-orthonormal systems is considered and the estimates for partial sum's majorant are established. © 2009 Bull. Georg. Natl. Acad. Sci.

Key words: block-orthonormal systems.

Let $\{\varphi_n\}$ be an orthonormal system from $L^2(0,1)$. By $\sigma(\{\varphi_n\})$ we denote the set of all sequences $\{a_n\}$ for which the series

$$\sum_{n=1}^{\infty} a_n \varphi_n(x) \tag{1}$$

converges almost everywhere on $(0,1)$.

K. Tandori [1] considered the set

$$\sigma_{\Omega} = \bigcap_{\{\varphi_n\} \in \Omega} \sigma(\{\varphi_n\}),$$

where Ω denotes the set of all orthonormal systems from $L^2(0,1)$. He studied the quantity

$$I(a_1, \dots, a_n) = \sup_{\{\varphi_n\} \in \Omega} \int_0^1 \max_{1 \leq l \leq n} \left| \sum_{i=1}^l a_i \varphi_i(x) \right|^2 dx \tag{2}$$

and by it for each sequence $\{a_i\}$ he has determined

$$\|\{a_i\}\| = \lim_{n \rightarrow \infty} I^{1/2}(a_1, \dots, a_n). \tag{3}$$

The quantity (3) always exists finite or infinite and it has all the properties of a norm.

Theorem (K. Tandori [1]). The class σ_{Ω} coincides with the set of all sequences $\{a_i\}$, for which the quantity

(3) is finite.

If $|a_1| \geq |a_2| \geq \dots$, for norm (3) is fulfilled (see also [2,3]):

$$c_1 \sum_{n=1}^{\infty} a_n^2 \log_2^2(n+1) \leq \|\{a_i\}\|^2 \leq c_2 \sum_{n=1}^{\infty} a_n^2 \log_2^2(n+1).$$

In the present paper we shall consider block-orthonormal systems and properties of partial sum's majorant of series with respect to block-orthonormal systems ([4,5]).

Definition([4]). Let $\{N_k\}$ be an increasing sequence of natural numbers, $\Delta_k = (N_k, N_{k+1}]$, ($k = 1, 2, \dots$) and $\{\varphi_n\}$ be a system of functions from $L^2(0,1)$. The system $\{\varphi_n\}$ will be called a Δ_k -orthonormal system (Δ_k -ONS) if:

- 1) $\|\varphi_n\|_2 = 1, \quad n = 1, 2, \dots;$
- 2) $(\varphi_i, \varphi_j) = 0$, for $i, j \in \Delta_k, \quad i \neq j, \quad k \geq 1$.

For each Δ_k -ONS $\{\varphi_n\}$ by $\sigma(\{\varphi_n\}, \Delta_k)$ we denote the set of all sequences $\{a_i\}$, for which the corresponding series (1) converges almost everywhere on $(0,1)$. $\Omega(\Delta_k)$ denotes the set of all block-orthonormal systems from $L^2(0,1)$. We denote:

$$I_{\Delta_k}(a_1, \dots, a_n) = \sup_{\{\varphi_n\} \in \Omega(\Delta_k)} \int_0^1 \max_{1 \leq l \leq n} \left| \sum_{i=1}^l a_i \varphi_i(x) \right|^2 dx$$

and

$$\|\{a_i\}, \Delta_k\| = \lim_{n \rightarrow \infty} I_{\Delta_k}^{1/2}(a_1, \dots, a_n). \tag{4}$$

It is clear that $\Omega \subset \Omega(\Delta_k)$, therefore

$$I(a_1, \dots, a_n) \leq I_{\Delta_k}(a_1, \dots, a_n)$$

and

$$\|\{a_i\}\| \leq \|\{a_i\}, \Delta_k\|.$$

Lemma 1. *The following inequalities are fulfilled:*

- 1) $I_{\Delta_k}(a_1, \dots, a_n) \geq \sum_{i=1}^n a_i^2;$
- 2) $I_{\Delta_k}(a_1, \dots, a_n) \leq I_{\Delta_k}(a_1, \dots, a_n, a_{n+1});$ (5)
- 3) $I_{\Delta_k}^{1/2}(a_1 + b_1, \dots, a_n + b_n) \leq I_{\Delta_k}^{1/2}(a_1, \dots, a_n) + I_{\Delta_k}^{1/2}(b_1, \dots, b_n);$
- 4) $I_{\Delta_k}(a_1, \dots, a_n, b_1, \dots, b_m) \geq I_{\Delta_k}(a_1, \dots, a_n) + I_{\Delta_k}\left(\underbrace{0, \dots, 0}_n, b_1, \dots, b_m\right);$
- 5) $I_{\Delta_k}(a_1, \dots, a_n) \leq I_{\Delta_k}(b_1, \dots, b_n)$, if $|a_i| \leq |b_i|, \quad (1 \leq i \leq n)$.

We note that from (5) the existence of limit (4) (finite or infinite) follows

Lemma 2. *Let for sequence $\{a_i\}$ the condition*

$$\|\{a_i\}, \Delta_k\| < \infty$$

be fulfilled. Then for every Δ_k -ONS $\{\varphi_n\}$ the corresponding series (1) converges by norm in $L^2(0,1)$.

Lemma 3. Let the condition

$$S_\varphi^*(\{a_i\}, x) = \sup_{1 \leq N < \infty} \left| \sum_{i=1}^N a_i \varphi_i(x) \right|$$

be fulfilled. Then for every sequence $\{a_i\}$ we have:

$$\sup_{\varphi \in \Omega(\Delta_k)} \|S_\varphi^*(\{a_i\}, x)\|_2 = \|\{a_i\}, \Delta_k\|.$$

Let

$$\sigma_{\Omega(\Delta_k)} = \bigcap_{\{\varphi_n\} \in \Omega(\Delta_k)} \sigma(\{\varphi_n\}, \Delta_k).$$

The following theorems are fulfilled.

Theorem 1. $\sigma_{\Omega(\Delta_k)}$ is the set of all sequences $\{a_i\}$, for which

$$\|S_\varphi^*(\{a_i\}, x)\|_2 \leq c,$$

where $\{\varphi_n\}$ is arbitrary Δ_k -ONS from $\Omega(\Delta_k)$. The constant C is dependent on sequence $\{a_i\}$.

Theorem 2. If positive nondecreasing sequence $\{\omega(n)\}$ satisfies conditions

$$\sum_{k=1}^{\infty} \frac{1}{\omega(N_k)} < \infty \text{ and } \log_2^2 n = O(\omega(n)), \quad (n \rightarrow \infty),$$

then for arbitrary numbers $|a_1| \geq |a_2| \geq |a_3| \geq \dots$ we have estimate

$$c_1 \sum_{n=1}^{\infty} a_n^2 \log_2^2 n \leq \sup_{\{\varphi_n\} \in \Omega} \|S_\varphi^*(\{a_i\}, x)\|_2 \leq \sup_{\{\varphi_n\} \in \Omega(\Delta_k)} \|S_\varphi^*(\{a_i\}, x)\|_2 \leq c_2 \sum_{n=1}^{\infty} a_n^2 \omega(n).$$

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ბლოკებში ორთოგონალური მწკრივების თითქმის ყველგან კრებადობისა და კერძო ჯამთა მაჟორანტის შესახებ

გ. ნადიბაიძე

ი. ჯაფარიშვილის სახ. თბილისის სახელმწიფო უნივერსიტეტი

(წარმოდგენილია აკადემიის წევრის ე. ნადარაიას მიერ)

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