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Mathematics

On Solving the External Three-Dimensional Dirichlet Problem for a Harmonic Function by the Probabilistic Method

Mamuli Zakradze*, Zaza Sanikidze*, Zaza Tabagari*

* *N. Muskhelishvili Institute of Computational Mathematics, Tbilisi*

(Presented by Academy Member Nikoloz Vakhania)

ABSTRACT. The algorithm of solving the external three-dimensional Dirichlet boundary value problem for a harmonic function by the probabilistic method is given. The algorithm consists of the following stages: 1) transition from an infinite domain to a finite domain by an inversion; 2) consideration of a new boundary problem on the basis of Kelvin's theorem for the obtained finite domain; 3) application of the probabilistic method to solving a new problem, which in turn is based on a computer simulation of the Wiener process; 4) definition of the solution of the statement problem for the infinite domain by the solution of the new problem. For illustration an example is considered.

***Key words**: three-dimensional Dirichlet problem, harmonic function, probabilistic method, computer simulation.*

Let D be an infinite domain in the Euclidian space E_3, bounded by one closed piecewise smooth surface S (i.e., $S=\bigcup_{j=1}^{m} S^j$), where each part S^j is a smooth surface). Besides, we assume: 1) equations of the parts S^j are given; 2) edges of the surface S are piecewise smooth contours; 3) for the surface S it is easy to show that a point $x=(x_1,x_2,x_3)\in E_3$ lies in $\overline{D}$ or not. For the Laplace equation we consider the Dirichlet boundary value problem.

Problem A. *Find a function* $u(x)\equiv u(x_1,x_2,x_3)\in C^2(D)\bigcap C(\overline{D})$ *satisfying the conditions:*

$$\Delta u(x)=0, \quad x\in D,$$

$$u(y)=g(y), \quad y\in S,$$

$$\lim u(x)=0, \text{ for } \quad x\to\infty,$$

where $\Delta=\sum_{i=1}^{3}\frac{\partial^2}{\partial x_i^2}$ *is the Laplace operator and* $g(y)\equiv g(y_1,y_2,y_3)$ *is a continuous function on S.*

It is known [1,2] that Problem A is correct, i.e., its solution exists, is unique and depends on data continuously. The third condition of Problem A is essential for the uniqueness of the solution. It should be noted that the laboriousness of solving problems sharply increases along with the dimension of the problems considered. Therefore, as a rule, one fails to develop standard methods for solving a wide class of multidimensional problems with the same high accuracy as in the one-dimensional case. In the example the exact solution of Problem A for a disk is written by one-dimensional Poisson's integral and in the case of sphere by two-dimensional Poisson's integral [2-4].

Since function harmonicity is invariable under the linear transformation of the Cartesian coordinates system, therefore without loss of generality we assume that the origin of coordinates $O(0,0,0)$ is inside a finite domain B bounded by the surface S.

In Problem A the domain D is infinite, therefore the direct application of the probabilistic method to its solving is impossible [5,6]. In order to solve Problem A by the probabilistic method, we convert from the infinite domain D to a finite domain D^* with a boundary S^* by means of the inversion [7]

$$\xi_i = x_i^0 + \frac{a^2(x_i - x_i^0)}{\left|x - x^0\right|^2}, \tag{1}$$

with respect to a sphere surface S_a. In (1) x^0 is a fixed inner point of the domain B, and a is the radius of the sphere S_a with the center at x^0. On the basis of the above denoted, for simplicity we can assume that $x^0=(0,0,0)$, $a=1$. Thus, in our case from (1) we have

$$\xi_i = \frac{x_i}{|x|^2}, \quad |x|^2 = x_1^2 + x_2^2 + x_3^2, \tag{2}$$

$$x_i = \frac{\xi_i}{|\xi|^2}, \quad |\xi|^2 = \xi_1^2 + \xi_2^2 + \xi_3^2. \tag{3}$$

It should be noted that under the inversions (2) and (3) the infinite domain D is transformed into the finite domain D^*, while the surface S is transformed into S^* and vice versa, i.e., $x \leftrightarrow \xi$, $x \in \overline{D}$ and $\xi \in \overline{D}^*$. In particular, the point $x = \infty$ goes to $O(0,0,0) \in D^*$. Therefore the functions $u(x)$ and $g(y)$ are transformed into the functions $u(\xi) \equiv u\left(\frac{\xi_1}{|\xi|^2}, \frac{\xi_2}{|\xi|^2}, \frac{\xi_3}{|\xi|^2}\right)$ and $g(\eta) \equiv g\left(\frac{\eta_1}{|\eta|^2}, \frac{\eta_2}{|\eta|^2}, \frac{\eta_3}{|\eta|^2}\right)$, respectively, where $\xi(\xi_1,\xi_2,\xi_3) \in D^*$, $\eta(\eta_1,\eta_2,\eta_3) \in S^*$.

It is known [2,7] that the function is not harmonic in the domain D^*. We can remove the noted defect of the inversion (2) if we apply Kelvin's theorem [2,7].

Theorem 1. *If a function $u(x_1,x_2,x_3)$ is harmonic in the domain D, then the function*

$$u^*(\xi_1,\xi_2,\xi_3) = \frac{1}{|\xi|} u\left(\frac{\xi_1}{|\xi|^2}, \frac{\xi_2}{|\xi|^2}, \frac{\xi_3}{|\xi|^2}\right) \tag{4}$$

is harmonic in the domain D^*, which is obtained from the domain D by the inversion (2).

Remark 1. It is shown that the point $\xi = 0$ is a removable singular point [2,7].

On the basis of Theorem 1 it is easy to see that actually the function $u^*(\xi_1,\xi_2,\xi_3)$ is the solution of the following boundary problem.

Problem A*. *Find a harmonic function $u^*(\xi)$ in D^* satisfying the conditions:*

$$\Delta u^*(\xi) = 0, \quad \xi \in D^*.$$

$$u^*(\eta) = g^*(\eta) \equiv \frac{1}{|\eta|} g(\eta), \quad \eta \in S^*,$$

where $g(\eta)$ is the given continuous function on S^*.

It is evident that the piecewise smooth surface S^* has the form $S^* = \bigcup_{j=1}^{m} (S^j)^*$, where the equations of the parts $(S^j)^*$ are defined by the equations of the parts S^j and inversion (3). Since the domain D^* is bounded by one closed piecewise smooth surface S^*, therefore for solving Problem A* we can apply the probabilistic method.

In particular, if we want to find the value of the solution $u(x)$ of Problem A at a point x $(x \in D)$, first of all we have to find the image ξ of x by means of (2), and then find the solution $u^*(\xi)$ to Problem A at the point ξ. Finally, on the basis of (4) we have

$$u(x) = |\xi| \cdot u^*(\xi_1, \xi_2, \xi_3) \equiv \frac{u^*(\xi)}{|x|}, \tag{5}$$

where $x \in D$, $\xi \in D^*$ and $\xi_i = \dfrac{x_i}{|x|^2}$ $(i = 1,2,3)$.

Thus, actually for definition of the value of the solution to Problem A at the point x we have the formula (5). It is easy to see that for the function $u(x)$, defined by (5), the conditions of Problem A are fulfilled.

The essence of solving Problem A* by the probabilistic method consists in the following.

It is known [5] that the probabilistic solution of Problem A* at the fixed point $\xi \in D^*$ has the form

$$u^*(\xi) = M_\xi g^*(\xi(t)), \tag{6}$$

where $M_\xi g^*(\xi(t))$ is the mathematical expectation of the values of the boundary function $g^*(\eta)$ at the random intersection points of the Wiener process and the boundary S^*; t is the moment of first exit of the Wiener process $\xi(t) = (\xi_1(t), \xi_2(t), \xi_3(t))$ from the domain D^*. It is assumed that the starting point of the Wiener process is always $\xi(t_0) = (\xi_1(t_0), \xi_2(t_0), \xi_3(t_0)) \in D^*$, where the value of the desired function is being determined. If the number N of the random points $\eta^i = (\eta_1^i, \eta_2^i, \eta_3^i) \in S^*$ $(i = 1,2,\ldots,N)$ is sufficiently large, then according to the law of large numbers, from (6) we have

$$u^*(\xi) \approx u_N^*(\xi) = \frac{1}{N}\sum_{i=1}^{N} g^*(\eta^i) \tag{7}$$

or $u^*(\xi) = \lim u_N^*(\xi)$ for $N \to \infty$, in the probabilistic sense. Thus, in the presence of the Wiener process we calculate the approximate value of the probabilistic solution of the Problem A* at the point $\xi \in D^*$ by formula (7).

For realization of the Wiener process we use the three-dimensional generator (see [6]), which gives three independent values $w_1(t), w_2(t), w_3(t)$. In the considered case the Wiener process is realized by computer simulation. In particular, for the computer simulation of the Wiener process we use the following recursion relations:

$$\begin{aligned} \xi_1(t_k) &= \xi_1(t_{k-1}) + w_1(t_k)/k\nu, \\ \xi_2(t_k) &= \xi_2(t_{k-1}) + w_2(t_k)/k\nu, \\ \xi_3(t_k) &= \xi_3(t_{k-1}) + w_3(t_k)/k\nu, \quad (k = 1,2,\ldots), \end{aligned} \tag{8}$$

with the help of which coordinates of a current point $(\xi_1(t_k), \xi_2(t_k), \xi_3(t_k))$ are being determined. In (8) $w_1(t_k), w_2(t_k), w_3(t_k)$ are three normally distributed independent random numbers for k-th step, with zero means and variances one; $k\nu$ is a number of the quantification and when $k\nu \to \infty$, then the discrete Wiener process approaches the continuous Wiener process. In the computer the random process is simulated at each step of the walk and continues until it crosses the boundary. In our case the noted random numbers are generated in the environment of the MATLAB system.

Example. The exterior of the unit sphere $S_1 : x_1^2 + x_2^2 + x_3^2 = 1$ with the center at the origin $O(0,0,0)$ is taken in the role of the domain D, where $x(x_1, x_2, x_3)$ is a current point of the surface S_1. $g(y) = 1/\left|y - \xi^0\right|$ is taken as boundary function, where $y(y_1, y_2, y_3) \in S\,(S \equiv S_1)$, $\xi^0 = (\xi_1^0, \xi_2^0, \xi_3^0) \in D^*$. It is evident that for the boundary function $g(y)$ the exact solution of Problem A is $u(x) = 1/\left|x - \xi^0\right|$. In the considered case on the basis of (2) and (3), for the boundary Problem A* we have $g^*(\eta) = (\alpha_1^2 + \alpha_2^2 + \alpha_3^2)^{-1/2}$, where $\alpha_i^2 = \left(\eta_i - x_i^0 / \left|x^0\right|^2\right)^2$ $(i = 1,2,3)$, $x^0 \leftrightarrow \xi^0$, $\eta = (\eta_1, \eta_2, \eta_3) \in S^*\,(S^* = S = S_1)$, $|\eta| = 1$.

While solving Problem A* to define the intersection points $\eta^i = (\eta_1^i, \eta_2^i, \eta_3^i)$ $(i = 1, 2, \ldots, N)$ of the Wiener process and of the surface S^*, we operate in the same way as is used in [6]. During the realization of the Wiener process, for each current point $\xi(t_k)$, defined from (8), its location with respect to the boundary S^* is checked, i.e., for the point $\xi(t_k)$ the value $d = |\xi(t_k)|^2 = \xi_1^2(t_k) + \xi_2^2(t_k) + \xi_3^2(t_k)$ is calculated and the conditions: $d = 1$, $d < 1$ or $d > 1$ are checked. If $d = 1$ then $\xi(t_k) \in S^*$ and $\eta^i = \xi(t_k)$. If $d < 1$, then $\xi(t_k) \in D^*$, and If $d > 1$, then $\xi(t_k) \notin \overline{D}^*$. Let $\xi(t_{k-1}) \in D^*$ for the moment $t = t_{k-1}$, and $\xi(t_k) \notin \overline{D}^*$ for the moment $t = t_k$. In this case, for an approximate definition of the point η^i, a parametric equation of a line l passing through the points $\xi(t_{k-1})$ and $\xi(t_k)$ is written in the first place. After this the intersection points η^* and η^{**} of the line l and of the surface S^* are defined. In the role of the point η^i from the points η^* and η^{**} a point is taken for which $|\xi(t_k) - \eta|$ is minimal. In numerical experiments $\xi^0 = (0,0,1/10)$ is taken.

In Table 1: N is the Wiener process realization number for the given points $x^j = (x_1^j, x_2^j, x_3^j) \in D$ $(j = 1,2)$; $\Delta^j = \left|u_1(x^j) - u(x^j)\right|$, where $u_1(x^j)$ is the approximate solution of problem A at the point x^j, which is defined by formula (5).

Table 1.

The results of experiments

	$x^1 = (1,1,1)$		$x^2 = (0,0,1.001)$	
	$k\nu = 100$	$k\nu = 200$	$k\nu = 100$	$k\nu = 200$
N	Δ^1	Δ^1	Δ^2	Δ^2
1000	2.94E-05	5.05E-05	8.44E-05	4.78E-05
4000	2.92E-05	3.79E-05	8.41E-05	4.64E-05
40000	2.71E-05	2.15E-05	8.39E-05	4.28E-05
200000	2.32E-05	1.79E-05	8.01E-05	3.85E-05

From Table 1 it is seen that $\Delta^j \to \infty$, when $N \to \infty$, in the probabilistic sense.

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მათემატიკა

ჰარმონიული ფუნქციისთვის დირიხლეს სამგანზომილებიანი გარე სასაზღვრო ამოცანის ალბათური მეთოდით ამოხსნის შესახებ

მ. ზაქრაძე*, ზ. სანიკიძე*, ზ. თაბაგარი*

* *ნ. მუსხელიშვილის გამოთვლითი მათემატიკის ინსტიტუტი, თბილისი*

(წარმოდგენილია აკადემიკოს ნ. ვახანიას მიერ)

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1) უსასრულო არიდან სასრულ არეზე გადასვლა ინვერსიის საშუალებით;

2) კელვინის თეორემის საფუძველზე ახალი ამოცანის განხილვა მოცემული სასრული არისათვის;

3) ახალი ამოცანის ამოხსნისათვის ალბათური მეთოდის გამოყენება, რომელსაც თავის მხრივ საფუძვლად უდევს ვინერის პროცესის კომპიუტერული მოდელირება.

4) უსასრულო არისათვის დასმული ამოცანის ამონახსნის განსაზღვრა ახალი ამოცანის ამონახსნის საშუალებით.

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