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Mathematics

On the Estimation of a Distribution Function by an Indirect Sample. I

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ABSTRACT. The problem of estimation of a distribution function is considered when the observer has access only to some indicator random values. Some basic asymptotic properties of the constructed estimates are studied. © *2010 Bull. Georg. Natl. Acad. Sci.*

Key words: *distribution function estimate, unbiased, consistency, asymptotic normality, estimate of time moments.*

Let $X_1, X_2, \ldots, X_n$ be a sample of independent observations of a nonnegative random value X with a distribution function $F(x)$. In problems of the theory of censored observations the sample values are pairs of random values $Y_i = (X_i \wedge t_i)$ and $Z_i = I(Y_i = X_i)$, $i = \overline{1,n}$, where t_i are given numbers ($t_i \neq t_j$ for $i \neq j$) or random values independent of X_i, $i = \overline{1,n}$. Throughout the paper $I(A)$ denotes the indicator of the set A.

We will consider here several different cases: the observer has access only to random values $\xi_i = I(X_i < t_i)$, $t_i = c_F \ \dfrac{2i-1}{2n}$, $i = \overline{1,n}$, $c_F = \inf\{x \geq 0: \ F(x) = 1\} < \infty$.

The problem consists in estimating distribution functions $F(x)$ by the sample $\xi_1, \xi_2, \ldots, \xi_n$. Such a problem arises, for example, in corrosion investigations (see [1] where an experiment connected with corrosion is described).

As estimate for $F(x)$ we consider an expression of the form

$$\hat{F}_n(x) = \begin{cases} 0, & x \leq 0, \\ F_{1n}(x) F_{2n}^{-1}(x), & 0 < x < c_F, \\ 1, & x \geq c_F, \end{cases} \tag{1}$$

$$F_{1n}(x) = \frac{1}{nh} \sum_{j=1}^{n} K\left(\frac{x - t_j}{h}\right) \xi_j,$$

$$F_{2n}(x) = \frac{1}{nh} \sum_{j=1}^{n} K\left(\frac{x - t_j}{h}\right),$$

where $K(x) \geq 0$ is some weight function (kernel) and $K(x) = K(-x)$, $-\infty < x < \infty$. $\{h = h(n)\}$ is a sequence of positive numbers converging to zero.

1. In this section, we give asymptotic unbiased and consistency conditions and theorem on a limiting distribution $\hat{F}_n(x)$.

Lemma. *Assume that*

1^0. $K(x)$ *is some distribution density with bounded variation. If* $nh \to \infty$, *then*

$$\frac{1}{nh}\sum_{j=1}^{n} K^{m_1-1}\left(\frac{x-t_j}{h}\right)F^{m_2-1}(t_j) = \frac{1}{c_F h}\int_0^{c_F} K^{m_1-1}\left(\frac{x-u}{h}\right)F^{m_2-1}(u)\ du + O\left(\frac{1}{nh}\right) \tag{2}$$

uniformly with respect to $x \in [0, c_F]$, m_1, m_2 *are natural numbers.*

Proof. Let $P(x)$ be a uniform distribution function on $[0, c_F]$ and $P_n(x)$ be an empirical distribution function of the "sample" $t_1, t_2, \ldots, t_n$, i.e. $P_n(x) = n^{-1}\sum_{j=1}^{n} I(t_j < x)$. It is obvious that

$$\sup_{0 \le x \le c_F} |P_n(x) - P(x)| = \sup_{0 \le x \le c_F} \left| \frac{1}{n}\left[n\ \frac{x}{c_F} + \frac{1}{2}\right] - \frac{x}{c_F} \right| \le \frac{1}{2n}. \tag{3}$$

We have

$$\frac{1}{nh}\sum_{i=1}^{n} K^{m_1-1}\left(\frac{x-t_i}{h}\right)F^{m_2-1}(t_i) - \frac{1}{c_F h}\int_0^{c_F} K^{m_1-1}\left(\frac{x-u}{h}\right)F^{m_2-1}(u)\ du =$$

$$= \frac{1}{h}\int_0^{c_F} K^{m_1-1}\left(\frac{x-u}{h}\right)F^{m_2-1}(u)\ d\left(P_n(u) - P(u)\right). \tag{4}$$

Applying the integration by parts of formula and taking (3) into account, we obtain (2) from (4).

Without loss of generality we assume below that the interval $[0, c_F] = [0, 1]$.

Theorem 1. *Let* $F(x)$ *be continuous and the conditions of the lemma be fulfilled. Then the estimate* (1) *is asymptotically unbiased and consistent at all points* $x \in [0,1]$. *Moreover,* $\hat{F}_n(x)$ *is distributed asymptotically normally, i.e.*

$$\sqrt{nh}\left(\hat{F}_n(x) - E\hat{F}_n(x)\right)\sigma^{-1}(x) \xrightarrow{d} N(0,1),$$
$$\sigma^2(x) = F(x)\left(1 - F(x)\right)\int K^2(u)\ du,$$

where d denotes convergence in distribution, and $N(0,1)$ *a random value having a normal distribution with mean 0 and variance* 1.

Proof. From the lemma we have

$$EF_{1n}(x) = \int_{\frac{x-1}{h}}^{\frac{x}{h}} K(t)F(x+ht)dt + O\left(\frac{1}{nh}\right), \quad F_{2n}(x) = \frac{1}{h}\int_0^1 K\left(\frac{x-u}{h}\right)du + O\left(\frac{1}{nh}\right), \tag{5}$$

and, as $n \to \infty$,

$$\frac{1}{h}\int_0^1 K\left(\frac{x-u}{h}\right)\ du \to F_2(x) = \begin{cases} 1, & x\in(0,1) \\ \frac{1}{2}, & x=0,\quad x=1 \end{cases}$$

$$\int_{\frac{x-1}{h}}^{\frac{x}{h}} K(t)F(x+th)dt \to F(x)\cdot F_2(x).$$

Hence it follows that $E\hat{F}_n(x)\to F(x)$, $x\in[0,1]$ as $n\to\infty$.

Analogously, it is not difficult to show that

$$Var\ \ \hat{F}_n(x) = \left[\frac{1}{nh^2}\int_0^1 K^2\left(\frac{x-u}{h}\right)F(u)\left(1-F(u)\right)\ du + O\left(\frac{1}{(nh)^2}\right)\right]F_{2n}^{-2}(x).$$

This readily implies that

$$nh\ \ Var\ \ \hat{F}_n(x) \sim \sigma^2(x) = F(x)\left(1-F(x)\right)\int K^2(u)\ du \tag{6}$$

as $x\in[0,1]$.

Thus $\hat{F}_n(x)$ is a consistent estimate for $F(x)$, $x\in[0,1]$, and therefore, $P\left\{\hat{F}_n(x_1)\le \hat{F}_n(x_2)\right\}\to 1$, $x_1<x_2$, x_1, $x_2\in[0,1]$.

Now we will establish that $\hat{F}_n(x)$ is distributed asymptotically normally. Since by virtue of (5), $F_{2n}(x)\to F_2(x)$, it remains for us to verify the condition of Lyapunov's Central Limit Theorem for $F_{1n}(x)$. Let us denote $\eta_i=\eta_i(x)=(nh)^{-1}K\left(\frac{x-t_i}{h}\right)\xi_i$ and show that

$$L_n = \sum_{j=1}^n E\left|\eta_j - E\eta_j\right|^{2+\delta}\left(Var\ \ F_{1n}(x)\right)^{-1-\frac{\delta}{2}} \to 0,\quad \delta>0. \tag{7}$$

We have

$$\sum_{j=1}^n E\left|\eta_j - E\eta_j\right|^{2+\delta} \le 2M^{1+\delta}(nh)^{-(2+\delta)}\sum_{j=1}^n K\left(\frac{x-t_j}{h}\right)F(t_j),\qquad M=\max_{x\in R}K(x).$$

Taking (2) into account, from this inequality we find

$$\sum_{j=1}^n E\left|\eta_j - E\eta_j\right|^{2+\delta} \le c_1(nh)^{-(1+\delta)}. \tag{8}$$

Using the relation (6) and the inequality (8) we obtain $L_n = O\left((nh)^{-\frac{\delta}{2}}\right)$, which means that (7) holds.

2. Uniform consistency. In this section, we define the conditions under which the estimate $\hat{F}_n(x)$ converges uniformly in probability (almost surely) to a true $F(x)$.

We introduce the Fourier transform of $K(x)$:

$$\varphi(t)=\int_{-\infty}^{\infty} e^{itx}K(x)\ dx$$

and assume that

2^0. $\varphi(t)$ is absolutely integrable. Following E. Parzen [2] $F_{1n}(x)$ can be written in the form

$$F_{1n}(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iu\frac{x}{h}}\varphi(u)\ \frac{1}{nh}\sum_{j=1}^{n}\xi_j e^{iu\frac{t_j}{h}}\ du .$$

Thus

$$F_{1n}(x)-EF_{1n}(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iu\frac{x}{h}}\varphi(u)\ \frac{1}{nh}\sum_{j=1}^{n}\left(\xi_j-F\left(t_j\right)\right)\ e^{iu\frac{t_j}{h}}\ du .$$

Denote

$$d_n=\sup_{x\in\Omega_n}\left|\hat{F}_n(x)-E\hat{F}_n(x)\right|,\quad \Omega_n=\left[h^\alpha,1-h^\alpha\right],\quad 0<\alpha<1 .$$

Theorem 2. *Let* $K(x)$ *satisfy conditions* 1^0 *and* 2^0.

(a) *Let* $F(x)$ *be continuous and* $n^{\frac{1}{2}}h_n\to\infty$, *then*

$$D_n=\sup_{x\in\Omega_n}\left|\hat{F}_n(x)-F(x)\right|\xrightarrow{P}0;$$

(b) *If* $\sum_{n=1}^{\infty}n^{-\frac{p}{2}}h^{-p}<\infty$, $p>2$, *then* $D_n\to 0$ *almost surely.*

Proof. We have

$$\sup_{x\in\Omega_n}\left(1-\frac{1}{h}\int_0^1 K\left(\frac{x-u}{h}\right)\ du\right)\le\int_{-\infty}^{-h^{\alpha-1}}K(u)\ du+\int_{h^{\alpha-1}}^{\infty}K(u)\ du\to 0 . \tag{9}$$

This and (5) imply

$$\sup_{x\in\Omega_n}\left|F_{2n}(x)-1\right|\to 0 \tag{10}$$

i.e., due to uniform convergence, for any $\varepsilon_0>0$, $0<\varepsilon_0<1$, and sufficiently large $n\ge n_0$ we have $F_{2n}(x)\ge 1-\varepsilon_0$ uniformly with respect to $x\in\Omega_n$.

Therefore

$$d_n\le\left(1-\varepsilon_0\right)^{-1}\sup_{x\in\Omega_n}\left|F_{1n}(x)-EF_{1n}(x)\right|\le\left(1-\varepsilon_0\right)^{-1}\cdot\frac{1}{2\pi}\int\left|\varphi(u)\right|\ \frac{1}{nh}\left|\sum_{j=1}^{n}\overline{\eta}_j e^{iu\frac{t_j}{h}}\right|\ du,\quad \overline{\eta}_j=\xi_i-F\left(t_j\right),$$

From here owing to Gelder's inequality, we have

$$d_n^p\le\left(1-\varepsilon_0\right)^{-p}\frac{1}{(2\pi)^p}\ \frac{1}{(nh)^p}\int\left|\varphi(u)\right|\ \left|\sum_{j=1}^{n}\overline{\eta}_j e^{iu\frac{t_j}{h}}\right|^p\ du\ \left(\int\left|\varphi(u)\right|\ du\right)^{\frac{p}{q}},\quad \frac{1}{p}+\frac{1}{q}=1,\quad p>2 .$$

Therefore

$$Ed_n^p \le c(\varepsilon, p, \varphi)\frac{1}{(nh)^p}\int |\varphi(u)| E\left|\sum_{j,k}\cos\left(\left(\frac{t_j - t_k}{h}\right)u\right)\ \bar{\eta}_j\bar{\eta}_k\right|^{\frac{p}{2}}\ du\,, \tag{11}$$

where

$$c(\varepsilon, p, \varphi) = (1-\varepsilon_0)^{-p}\ \frac{1}{(2\pi)^p}\ \left(\int |\varphi(u)|\ \ du\right)^{\frac{p}{q}}.$$

Denote

$$A(u) = \sum_{j,k}\cos\left(\left(\frac{t_j - t_k}{h}\right)u\right)\ \bar{\eta}_j\bar{\eta}_k\ .$$

Then from (11) we can write

$$Ed_n^p \le 2^{\frac{p}{2}-1} c(\varepsilon_0, p, \varphi)\frac{1}{(nh)^p}\left[\int |\varphi(u)|\ \ |EA(u)|^{\frac{p}{2}}\ \ du + \int |\varphi(u)|\ \ E|A(u) - EA(u)|^{\frac{p}{2}}\ \ du\right]. \tag{12}$$

Further, using Whittle's inequality [3] for moments of quadratic form, we obtain

$$E|A(u) - EA(u)|^{\frac{p}{2}} \le 2^{\frac{3}{2}p} c\left(\frac{p}{2}\right)[c(p)]^{\frac{1}{2}}\ \left(\sum_{i,j}\cos^2\left(\left(\frac{t_j - t_k}{h}\right)u\right)\ \gamma_j^2(p)\lambda^2(p)\right)^{\frac{p}{4}},$$

where

$$\gamma_k(p) = \left(E|\bar{\eta}_k|^p\right)^{\frac{1}{p}} \le 1, \qquad c(s) = \frac{2^{\frac{s}{2}}}{\sqrt{\pi}}\ \Gamma\left(\frac{s+1}{2}\right).$$

From here follows

$$E|A(u) - EA(u)|^{\frac{p}{2}} = O\left(n^{\frac{p}{2}}\right), \tag{13}$$

uniformly with respect to $u \in (-\infty, \infty)$, and also clear that,

$$|EA(u)|^{\frac{p}{2}} = O\left(n^{\frac{p}{2}}\right), \tag{14}$$

uniformly with respect to $u \in (-\infty, \infty)$. After combining the relations (12), (13) and (14), we obtain

$$Ed_n^p = O\left(\frac{1}{\left(\sqrt{n}\ \ h\right)^p}\right), \qquad p > 2\,.$$

Therefore

$$P\left\{\sup_{x\in\Omega_n}\left|\hat{F}_n(x) - E\hat{F}_n(x)\right| \ge \varepsilon\right\} \le \frac{c_3}{\varepsilon^p\left(\sqrt{n}\ \ h\right)^p}\,. \tag{15}$$

Further we obtain

$$\sup_{x\in\Omega_n}\left|E\hat{F}_n(x) - F(x)\right| \le \frac{1}{1-\varepsilon_0}\ \left(\sup_{x\in\Omega_n}\left|EF_{1n}(x) - F(x)\right| + \sup_{x\in\Omega_n}\left|1 - F_{2n}(x)\right|\right). \tag{16}$$

The second summand in the right-hand part of (16) tends, by virtue of (10), to zero, while the first summand is estimated as follows:

$$\sup_{x\in\Omega_n}\left|EF_{1n}(x)-F(x)\right|\le S_{1n}+S_{2n}+O\left(\frac{1}{nh}\right), \tag{17}$$

$$S_{1n}=\sup_{0\le x\le 1}\left|\frac{1}{h}\int_0^1\bigl(F(y)-F(x)\bigr)K\left(\frac{x-y}{h}\right)dy\right|,$$

$$S_{2n}=\sup_{x\in\Omega_n}\left(1-\frac{1}{h}\int_0^1 K\left(\frac{x-y}{h}\right)dy\right),$$

and, by virtue of (9),

$$S_{2n}\to 0. \tag{18}$$

Now let us consider S_{1n}. Note that

$$S_{1n}\le\sup_{0\le x\le 1}\int_0^1\left|F(y)-F(x)\right|\frac{1}{h}K\left(\frac{x-y}{h}\right)dy=\sup_{0\le x\le 1}\int_{x-1}^{x}\left|F(x-u)-F(x)\right|\frac{1}{h}K\left(\frac{u}{h}\right)du\le$$

$$\le\sup_{0\le x\le 1}\int_{-\infty}^{\infty}\left|F(x-u)-F(x)\right|\frac{1}{h}K\left(\frac{u}{h}\right)du. \tag{19}$$

Assume that $\delta>0$ and divide the integration domain in (19) into two domains $|u|\le\delta$ and $|u|>\delta$. Then

$$S_{1n}\le\sup_{0\le x\le 1}\int_{|u|\le\delta}\left|F(x-u)-F(x)\right|\frac{1}{h}K\left(\frac{u}{h}\right)du+\sup_{0\le x\le 1}\int_{|u|>\delta}\left|F(x-u)-F(x)\right|\frac{1}{h}K\left(\frac{u}{h}\right)du\le$$

$$\le\sup_{x\in R}\sup_{|u|\le\delta}\left|F(x-u)-F(x)\right|+2\int_{|u|\ge\frac{\delta}{h}}K(u)\,du. \tag{20}$$

By a choice of $\delta>0$ the first summand in the right-hand part of (20) can be made arbitrarily small. After choosing $\delta>0$ and making n tend to infinity, we obtain that the second summand tends to zero.

Thus

$$\lim_{n\to\infty}S_{1n}=0. \tag{21}$$

Finally, the proof of the theorem follows from the relations (15)-(18) and (21).

Remarks.

1) If $K(x)=0$, $|x|\ge 1$ and $\alpha=1$, i.e., $\Omega_n=[h,1-h]$, then $S_{2n}=0$.

2) Under the conditions of Theorem 2,

$$\sup_{x\in[a,b]}\left|\hat{F}_n(x)-F(x)\right|\to 0$$

in probability (almost surely) for any fixed interval $[a,b]\subset[0,1]$ since there exists n_0 such that $[a,b]\subset\Omega_n$, $n\ge n_0$.

Let us assume that $h=n^{-\gamma}$, $\gamma>0$. The conditions of Theorem 2 are fulfilled:

$$n^{\frac{1}{2}} h_n \to \infty \text{ if } \gamma < \frac{1}{2}$$

and

$$\sum_{n=1}^{\infty} n^{-\frac{p}{2}} h_n^{-p} < \infty \text{ if } 0 < \gamma < \frac{p-2}{2p}, \quad p > 2 .$$

3. Estimation of moments. In the considered problem there naturally arises the question of estimation of the integral functional of $F(x)$, for example, of moments μ_m, $m \geq 1$:

$$\mu_m = m \int_0^1 t^{m-1} \left(1 - F(t)\right) \; dt .$$

As estimates for μ_m we will consider the statistics

$$\hat{\mu}_{nm} = 1 - \frac{m}{n} \sum_{j=1}^{n} \xi_j \;\; \frac{1}{h} \int_h^{1-h} t^{m-1} K\left(\frac{t - t_j}{h}\right) F_{2n}^{-1}(t) \; dt .$$

Theorem 3. *Let $K(x)$ satisfy condition 1^0 and, in addition to this, $K(x) = 0$ outside the interval $[-1,1]$. If $nh \to \infty$ as $n \to \infty$, then $\hat{\mu}_{nm}$ is an asymptotically unbiased, consistent estimate for μ_m and, moreover,*

$$\frac{\sqrt{n}\left(\hat{\mu}_{nm} - E\hat{\mu}_{nm}\right)}{\sigma} \xrightarrow{d} N(0,1), \qquad \sigma^2 = m^2 \int_0^1 t^{2m-2} F(t)\left(1 - F(t)\right) \; dt .$$

Proof. Since $K(x)$ has $[-1,1]$ as a carrier, from (5) it follows that

$$F_{2n}(n) = 1 + O\left(\frac{1}{nh}\right)$$

uniformly with respect to $x \in [h, 1-h]$.

From this and the lemma we have

$$E\hat{\mu}_{nm} = 1 - \frac{m}{n} \sum_{j=1}^{n} F(t_j) \frac{1}{h} \int_h^{1-h} t^{m-1} K\left(\frac{t - t_j}{h}\right) F_{2n}^{-1}(t) dt = 1 - m \int_h^{1-h} \left[\frac{1}{h} \int_0^1 K\left(\frac{t-u}{h}\right) F(u) du\right] t^{m-1} dt + O\left(\frac{1}{nh}\right) =$$

$$= 1 - m \int_h^{1-h} \left(\int_{-1}^{1} K(v) F(t + vh) \;\; dv\right) t^{m-1} dt + O\left(\frac{1}{nh}\right) = 1 - m \int_0^1 t^{m-1} \left[\int_{-1}^{1} K(u) F(t + vh) dv\right] dt + O(h) + O\left(\frac{1}{nh}\right). \quad (22)$$

By Lebesgue's theorem on majorized convergence, from (22) it follows that

$$E\hat{\mu}_{nm} \to 1 - m \int_0^1 F(t) \;\; t^{m-1} \;\; dt = m \int_0^1 t^{m-1} \left(1 - F(t)\right) \; dt = \mu_m, \qquad m \geq 1 . \quad (23)$$

Therefore $\hat{\mu}_{nm}$ is an unsymptotically unbiased estimate for μ_m.

Further, analogously to (22) it can be shown that

$$Var \;\; \hat{\mu}_{nm} = \frac{m^2}{n} \int_0^1 F(t)\left(1 - F(t)\right) \;\; t^{2m-2} \left[\mathcal{K}\left(\frac{1-t}{h} - 1\right) - \mathcal{K}\left(1 - \frac{t}{h}\right)\right]^2 \;\; dt + O\left(\frac{h}{n}\right) + O\left(\frac{1}{(nh)^2}\right),$$

where

$$\mathcal{K}(v) = \int_{-\infty}^{v} K(u)\ du .$$

By the same Lebesgue's theorem we see that

$$n\ \ Var\ \ \hat{\mu}_{nm} \sim \sigma^2 = m^2 \int_0^1 t^{2m-2} F(t)\big(1 - F(t)\big)\ \ dt . \tag{24}$$

Therefore (23) and (24) imply that $\hat{\mu}_{nm} \xrightarrow{P} \mu_m$.

To complete the proof of the theorem it remains to show that the statistics $\sqrt{n}\left(\hat{\mu}_{nm} - E\hat{\mu}_{nm}\right)$ are asymptotically distributed normally with mean 0 and variance σ^2. For this it suffices to show that the Lyapunov fraction $L_n \to 0$. Indeed,

$$L_n = n^{-(2+\delta)} m^{2+\delta} \sum_{j=1}^{n} \left|\xi_j - F(t_j)\right|^{2+\delta} \left| \frac{1}{h} \int_h^{1-h} t^{m-1} K\left(\frac{t - t_j}{h}\right) F_{2n}^{-1}\ \ dt \right|^{2+\delta} \left(Var\,\hat{\mu}_{nm}\right)^{-\left(1+\frac{\delta}{2}\right)} \le$$

$$\le c_6 n^{-(2+\delta)} \sum_{j=1}^{n} \left|\xi_j - F(t_j)\right|^{2+\delta} \left(Var\,\hat{\mu}_{nm}\right)^{-\left(1+\frac{\delta}{2}\right)} \le c_7 n^{-1-\delta} \left(Var\,\hat{\mu}_{nm}\right)^{-\left(1+\frac{\delta}{2}\right)} = O\left(n^{-\frac{\delta}{2}}\right).$$

The theorem is proved.

მათემატიკა

განაწილების ფუნქციის შეფასება არაპირდაპირი შერჩევით. I

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