Physics

Calculation of the Gravitoelectromagnetism Force for the O’Hanlon-Tupper Spacetime

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ABSTRACT. By using the threading splitting concepts for a time dependent spacetime, the time dependent quasi-Maxwell equations in terms of the gravitoelectromagnetism fields are discussed. The motion of a test particle in the O’Hanlon-Tupper spacetime by applying the Hamilton-Jacobi method and the quasi-Maxwell equations is studied. Also, the gravitoelectromagnetism force in this spacetime is calculated. © 2013 Bull. Georg. Natl. Acad. Sci.

Key words: O’Hanlon-Tupper spacetime, quasi-Maxwell equations, particle trajectory, gravitoelectromagnetism force.

1. Introduction


In threading point of view, splitting of spacetime with unit tangent vector field may be interpreted as the world-lines of a family of observers, and it defines a local time direction plus a local space through its orthogonal subspace in tangent space. Let \((M, g_{\alpha\beta})\) be a 4-dim manifold of a stationary spacetime. The Greek indices run from 0 to 3 while the Latin indices take the values 1 to 3. We can construct a 3-dim orbit manifold as \(\mathcal{M} = \frac{M}{G}\) with projected metric tensor \(\gamma_{ij}\) by the smooth map \(\Sigma: M \rightarrow \mathcal{M}\), where \(\Sigma(p)\) denotes the orbit of the timelike Killing vector \(\frac{\partial}{\partial t}\) at the point \(p \in M\) and \(G\) is 1-dim group of transformations generated by timelike Killing vector of the spacetime under con-
sideration [7,8]. The threading decomposition leads to the following line element [2,8]:
\[ ds^2 = g_{ij} dx^i dx^j = h \left( dt - g_i dx^i \right)^2 - g_{ij} dx^i dx^j , \]  
\text{(1)}
where \( \gamma_{ij} = -g_{ij} + h \gamma_{ij} \) in which \( g_{ij} = -\frac{\delta_{ij}}{h} \) and \( h = g_{00} \). If we apply the time dependent \( \gamma_{ij} \) as the metric, then the vacuum Einstein field equations may be written as the time dependent quasi-Maxwell equations (c = G = 1) [6,9]:
\[ *\nabla \cdot *E_g = *E_g^2 + \frac{1}{2} *H_g^2 = \frac{\partial D}{\partial t} - d, \]  
\text{(2)}
\[ *\nabla \times *H_g = 2 \left( *S_g + *M \right), \]  
\text{(3)}
\[ *K_g = -*\nabla ( *E_g + *E_g^2 ) + \frac{1}{2} \left( *H_g + *H_g^2 \right) \]  
\text{(4)}
\[ + 2D_k D^l \frac{\partial}{\partial t} + \sqrt{\gamma} \varepsilon_{nkj} D^l_n H_{k}^l, \]
where \( \frac{\partial}{\partial t} = \frac{1}{\sqrt{h}} \frac{\partial}{\partial t} \), \( \gamma = \det(\gamma_{ij}) \) and \( d = D_k D^k \), such that
\[ D_k = \frac{1}{2} \frac{\partial}{\partial t}, D^l = \frac{1}{2} \frac{\partial}{\partial t}, D = \gamma D_0 = \frac{\partial}{\partial t}. \]  
\text{(5)}

The symbols () and [] represent the commutation and anticommutation over indices and the 3-dim Levi-Civita tensor \( \varepsilon_{ijk} \) is antisymmetric under interchange of any pair of indices such that \( \varepsilon_{123} = \varepsilon_{123} = 1 \) [2]. Also, divergence and curl of an arbitrary vector in a 3-space with metric \( \gamma_{ij} \) are defined as
\[ *\nabla \cdot A = \frac{1}{\sqrt{\gamma}} \left( \sqrt{\gamma} A^i \right)_i \quad \text{and} \quad \left( *\nabla \times A \right)^i = \varepsilon^{ijk} A_{[k]} \]  
such that \( \gamma_{ij} = \partial_i + g_{ij} \frac{\partial}{\partial t} \). In equation (4), the 3-dim starry Ricci tensor \( *K_{ij} \) is constructed from 3-dim starry Christoffel symbols as \( *K_{ij} = \lambda_{ijk}^k - \lambda_{ik}^k + \)  
\[ + \lambda_{ij}^k \lambda_{ik}^k - \lambda_{ik}^k \lambda_{ij}^k \]  
where \( \lambda_{jk} = \frac{1}{2} \gamma^{ij} \left( \gamma_{jk}^i + \right) \) and the starry covariant derivatives of an arbitrary 3-vector are given by \( *\nabla_j A_i = \)  
\[ = A_{ij} - \chi_{ij} A_k^k \quad \text{and} \quad *\nabla_j A_i = \lambda_{ij}^k + \lambda_{ik}^j A_k. \]  
The time dependent gravitoelectromagnetism fields are defined in terms of the gravoelectric potential \( \varphi = \ln \sqrt{h} \) and the gravomagnetic vector potential \( g = (g_1, g_2, g_3) \) as follows
\[ *E_g = - *\nabla \varphi - \frac{\partial g_k}{\partial t}; \quad *E_g = - \varphi_t - \frac{\partial g_k}{\partial t}, \]  
\text{(6)}
\[ *\frac{H_g}{\sqrt{h}} = *\nabla \times \gamma; \quad *\frac{H_g}{\sqrt{h}} = \frac{\varepsilon^{ijk}}{2 \sqrt{\gamma}} [\gamma_i (\gamma_{jk})]. \]  
\text{(7)}

The gravitoelectromagnetism refers to a set of analogies between Maxwell equations and a reformulation of Einstein field equations in general relativity [10,11]. The vectors \( *S_g = *E_g \times *H_g \) and \( *M \) have components as \( *S_g^i = \varepsilon^{ijk} *E_g^j *H_g^k \) and \( *M^i = - *\nabla_j D^j + *\partial^j D \) while \( *\nabla_j D^j = D^j + \)  
\[ + \gamma_{ij} D^j + \lambda_{ik} D^k \]  
and \( *\partial^j = \gamma^j + \partial^j \). For more details about applications of gravitoelectromagnetism fields, see [12-14].

2. Motion of a test particle in the O’Hanlon-Tupper spacetime

2.1. Calculation of the trajectory

We consider the O’Hanlon-Tupper metric in Cartesian coordinates as follows [15]:
\[ ds^2 = dt^2 - \gamma_{ij} \left( dx^2 + dy^2 + dz^2 \right), \]  
\text{(8)}
where \( n \) is an unknown constant. At first, it is easy to see that all components of gravitoelectromagnetism fields and starry Christoffel symbols vanish. Therefore, the time-dependent quasi-Maxwell equations reduce to

The solutions of these equations for the metric (8) are
\[ n = \frac{2}{3} \text{ & } 2. \] (11)

As is known [16], the solution \( n = 2 \) corresponds to the Chitre-Hartle spacetime. The Chitre-Hartle metric was first introduced as a model background for the study of scalar particle creation in homogeneous and isotropic spacetimes [17]. The physical significance of this spacetime was first pointed out by Fischetti et al. [18], who showed it to be a cosmological solution of the Einstein equations modified to include one-loop quantum gravitational corrections arising from the quantum trace anomaly.

In continuation, we determine the trajectory of a test particle of mass \( m \) that is moving in the O’Hanlon-Tupper spacetime by using the Hamilton-Jacobi equation [19-21]. Therefore, this equation is of the form
\[ t^n \left( \frac{\partial S}{\partial t} \right)^2 - \left( \frac{\partial S}{\partial x} \right)^2 - \left( \frac{\partial S}{\partial y} \right)^2 - \left( \frac{\partial S}{\partial z} \right)^2 - m^2 r^n = 0. \] (12)

To solve this partial differential equation, we use the method of separation of variables for the Hamilton-Jacobi function as below
\[ S(x, y, z, t) = L_x x + L_y y + L_z z + T(t), \] (13)
where \( L_x, L_y \) and \( L_z \) are arbitrary constants and can be identified respectively as the angular momentum components of test particle along \( x, y \) and \( z \)-directions. Substituting the ansatz (13) into the Hamilton-Jacobi equation, gives the following expression for the function \( T \) as
\[ T = \sigma \sqrt{\frac{\sqrt{m^2 r^2 + L_z^2} - L \ln t}{m^2 r^2 + L_z^2 + 2L_z^2}}, \quad n = 2 \] (14)
in which \( L^2 = L_x^2 + L_y^2 + L_z^2 \) and \( \sigma = \pm 1 \). Let us now obtain the trajectory of test particle by considering the following relations [19-21]:
\[ \frac{\partial S}{\partial L_x} = \text{constant}, \quad \frac{\partial S}{\partial L_y} = \text{constant}, \quad \frac{\partial S}{\partial L_z} = \text{constant}. \] (15)

Finally, after calculating and simplifying, the set of equations (15) changes to the following relations
\[ \frac{x}{L_x} = \frac{y}{L_y} = \frac{z}{L_z} = \frac{3L \sqrt{m^2 r^2 + L_z^2}}{m^2 r^2 + L_z^2 + 2L_z^2}, \quad n = \frac{2}{3}, \] (16)
we have taken the constants in equations (15) to be zero without any loss of generality. Therefore, the trajectory of particle have been calculated.

### 2.2. Calculation of the gravitoelectromagnetism force

In a spacetime with time dependent metric (1), the gravitoelectromagnetism force acting on a test particle whose mass \( m \) due to time dependent gravitoelectromagnetism fields as measured by threading observers is described by the following
\[ *F_g = \frac{d}{dt} *p - \frac{m}{\sqrt{1 - v^2}} \left( *E_g + *v \times *H_g + f \right), \] (17)
where \( p^i = \frac{m^i v^i}{\sqrt{1 - v^2}} \) such that \( v^2 = g_{ij} v^i v^j \) in which \( v^i = \frac{v^i}{\sqrt{h(1 - g_{ij} v^j)}} \) while \( v^i = \frac{dx^i}{dt} \). Also, the starry total derivative with respect to time is defined as follows

\[
* \frac{d}{dt} = \frac{\partial}{\partial t} + v^i \frac{\partial}{\partial v^i}.
\]  

(18)

In the equation (17), the last term is defined as

\[
f^i = \kappa_{jk} v^j v^k - 2D_i v^k.
\]  

(19)

As before, with employing the equations (16), we can obtain

\[
v = -\kappa_{jk} v^j v^k, \quad \sigma_L = L_x = L_y = L_z.
\]  

(20)

in which \( L = L_x i + L_y j + L_z k \). In the next step, by using the previous result, we lead to

\[
m \sqrt{1 - v^2} = \frac{n}{2} \sqrt{m^2 t^n + L^2}.
\]  

(21)

The above radicand has no real extremals. Hence, there are no bound states and the particle cannot be trapped by the extended object with the O’Hanlon-Tupper geometry. To continue our analysis, we need to calculate the last term of the equation (17). Thus, we have

\[
f = \frac{n \sigma L}{2} \sqrt{m^2 t^n + L^2}.
\]  

(22)

With the help of equations (20-22), after some calculations, we finally obtain

\[F_g = 0.\]

(23)

3. Conclusions

The classical motion of a test particle in the O’Hanlon-Tupper spacetime has been studied. We proved that the test particle cannot be trapped by this gravitational field. Also, it was shown that the gravito-electromagnetism force acting on test particle in this spacetime vanishes.

4. Acknowledgement

This research has been supported in part by Islamic Azad University-Kashan Branch.
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Received August, 2011