საქართველოს მეცნიერებათა ეროვნული აკადემიის მოამბე, ტ. 8, №1, 2014
BULLETIN OF THE GEORGIAN NATIONAL ACADEMY OF SCIENCES, vol. 8, no. 1, 2014

Physics

Classical Motion of a Relativistic Test Particle in the Static Cylindrically Symmetric Metric

Morteza Yavari

Department of Physics, Islamic Azad University, Kashan Branch, Kashan, Iran

(Presented by Academy Member Anzor Khelashvili)

ABSTRACT. In the paper, the gravitoelectric and gravitomagnetic fields are discussed in the threading formalism. The motion of a relativistic test particle in the static cylindrically symmetric metric is studied by applying the Hamilton-Jacobi method. In threading formalism the gravitoelectromagnetic force in this spacetime is also calculated. *© 2014 Bull. Georg. Natl. Acad. Sci.*

Keywords: *Static cylindrically symmetric metric, particle trajectory, gravitoelectromagnetism force.*

1. Introduction

The slicing and threading points of view today are introduced, respectively, by Misner, Thorne and Wheeler [1] in 1973 and Landau and Lifshitz [2] in 1975. Both points of view can be traced back when the Landau and Lifshitz [3] in 1941 introduced the threading point of view splitting of the spacetime metric. After them, Lichnerowicz [4] introduced the beginnings of slicing point of view. The slicing point of view is commonly referred as 3+1 or ADM formalism and also term 1+3 formalism has been suggested for the threading point of view. For more details about these formalisms, see reference [5]. In threading point of view, splitting of spacetime introduced by a family of timelike congruences with unit tangent vector field may be interpreted as the world-lines of a family of observers, and it defines a local time direction plus a local space through its orthogonal subspace in the tangent space. Let $(\mathrm{M}, g_{\alpha\beta})$ be a 4-dimensional manifold of a stationary spacetime, while the Greek indices run from 0 to 3 while the Latin indices take the values 1 to 3. We can construct a 3-dimensional orbit manifold as $\bar{\mathrm{M}} = \frac{\mathrm{M}}{\mathrm{G}}$ with projected metric tensor γ_{ij} by the smooth map $\Sigma : \mathrm{M} \to \bar{\mathrm{M}}$ where $\Sigma(p)$ denotes the orbit of the timelike Killing vector $\frac{\partial}{\partial t}$ at the point $p \in \mathrm{M}$ and G is 1-dimensional group of transformations generated by timelike Killing vector of the spacetime under consideration, [5,6]. The threading decomposition leads to the following line element [2,6,7]:

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta = h\left(dt - g_i dx^i\right)^2 - \gamma_{ij}dx^i dx^j, \tag{1}$$

where $\gamma_{ij} = -g_{ij} + hg_i g_j$, in which $g_i = -\frac{g_{0i}}{h}$ and $h = g_{00}$. In a spacetime with the time dependent metric (1), the gravitoelectromagnetic force acting on a relativistic test particle whose mass m due to time dependent gravitoelectromagnetic fields as measured by threading observers is described by the following equation, we use the gravitational units with c=1 [8,9]:

$$^*\mathbf{F} = \frac{^*d\,^*\mathbf{p}}{dt} - \frac{m}{\sqrt{1-{}^*v^2}}\left\{{}^*\mathbf{E} + {}^*\boldsymbol{v}\times{}^*\mathbf{B} + {}^*\mathbf{M}\right\}, \tag{2}$$

where ${}^*p^i = \frac{m\,{}^*v^i}{\sqrt{1-{}^*v^2}}$ such that ${}^*v^2 = \gamma_{ij}\,{}^*v^i\,{}^*v^j$ in which ${}^*v^i = \frac{v^i}{\sqrt{h}\left(1-g_k v^k\right)}$ with $v^i = \frac{dx^i}{dt}$ and starry total derivative with respect to time is defined as $\frac{^*d}{dt} = \frac{^*\partial}{\partial t} + {}^*v^i\,{}^*\partial_i$ where $\frac{^*\partial}{\partial t} = \frac{1}{\sqrt{h}}\frac{\partial}{\partial t}$ and $_{*i} = {}^*\partial_i = \partial_i + g_i\frac{\partial}{\partial t}$. We recall that the vector $\mathbf{C} = \mathbf{A}\times\mathbf{B}$ have the components as $C^i = \frac{\varepsilon^{ijk}}{\sqrt{\gamma}}A_j B_k$ in which $\gamma = \det(\gamma_{ij})$ and 3-dimensional Levi-Civita tensor ε_{ijk} is antisymmetric in any exchange of indices while $\varepsilon_{123} = \varepsilon^{123} = 1$, [2].

The gravitoelectromagnetic refers to a set of analogies between Maxwell equations and a reformulation of the Einstein field equations in general relativity [10,11].

In equation (2), the last term is defined as ${}^*M^i = -\left({}^*\lambda^i_{jk}\,{}^*v^j + 2D^i_k\right){}^*v^k$, where the 3-dimensional starry Christoffel symbols are defined with the following form

$${}^*\lambda^i_{jk} = \frac{1}{2}\gamma^{il}\left(\gamma_{jl*k} + \gamma_{kl*j} - \gamma_{jk*l}\right), \tag{3}$$

and deformation rates of the reference frame with respect to the observer are represented by tensors $D_{ij} = \frac{1}{2}\frac{^*\partial\gamma_{ij}}{\partial t}$ and $D^{ij} = -\frac{1}{2}\frac{^*\partial\gamma^{ij}}{\partial t}$. Finally, the time dependent gravitoelectromagnetic fields are defined in terms of gravoelectric potential $\Phi = \ln\sqrt{h}$ and the gravomagnetic vector potential $\mathbf{g} = \left(g_1, g_2, g_3\right)$ as follows

$${}^*\mathbf{E} = -{}^*\nabla\Phi - \frac{\partial\mathbf{g}}{\partial t},\quad {}^*E_i = -\Phi_{*i} - \frac{\partial g_i}{\partial t}, \tag{4}$$

$$\frac{^*\mathbf{B}}{\sqrt{h}} = {}^*\nabla\times\mathbf{g},\quad \frac{^*B^i}{\sqrt{h}} = \frac{\varepsilon^{ijk}}{2\sqrt{\gamma}}g_{[k*j]}, \tag{5}$$

where curl of an arbitrary vector in a 3-space with metric γ_{ij} is defined by $\left({}^*\nabla\times\mathbf{A}\right)^i = \frac{\varepsilon^{ijk}}{2\sqrt{\gamma}}A_{[k*j]}$ while the symbol [ ] represents the anticommutation over indices.

2. Classical motion of a relativistic test particle in the cylindrically symmetric metric

2.1. Calculation of the trajectory

We start with a static metric in the cylindrical coordinates (t, r, ϕ, z) given by, [12]:

$$ds^2 = e^{2k-2u}dt^2 - e^{2k-2u}dr^2 - w^2e^{-2u}d\phi^2 - e^{2u}dz^2, \tag{6}$$

where k, u and w are unknown functions of r only. At first, we determine the trajectory of a particle of mass m moving in this spacetime by using the Hamilton-Jacobi equation [13-15]. Therefore, this equation is of the form

$$\left(\frac{\partial S}{\partial t}\right)^2 - \left(\frac{\partial S}{\partial r}\right)^2 - \frac{e^{2k}}{w^2}\left(\frac{\partial S}{\partial \phi}\right)^2 - e^{2k-4u}\left(\frac{\partial S}{\partial z}\right)^2 - m^2e^{2k-2u} = 0. \tag{7}$$

We now use the method of separation of variables for the Hamilton-Jacobi function as

$$S(t, r, \phi, z) = -Et + \xi(r) + a\phi + bz, \tag{8}$$

where E, a and b are arbitrary constants and can be identified respectively as energy and angular momentum of test particle along ϕ and z-directions. Next, with substituting the relation (8) in the Hamilton-Jacobi equation, the unknown function ξ is given by

$$\xi = \varepsilon\int\sqrt{E^2 - \lambda e^{2k}}\,dr, \tag{9}$$

here $\lambda = m^2e^{-2u} + b^2e^{-4u} + (\frac{a}{w})^2$ and $\varepsilon = \pm 1$ stands for the sign changing whenever r passes through a zero of the integrand in the equation (9). The equations for the trajectory can be obtained by considering the following conditions [13-15]:

$$\frac{\partial S}{\partial E} = \text{constant}, \frac{\partial S}{\partial a} = \text{constant}, \frac{\partial S}{\partial b} = \text{constant}, \tag{10}$$

without any loss of generality one can consider the above constants to be zero. Hence, the equations (10) respectively convert to the following relations

$$t = \varepsilon E\int\frac{dr}{\sqrt{E^2 - \lambda e^{2k}}}, \tag{11}$$

$$\phi = \varepsilon a\int\frac{e^{2k}dr}{w^2\sqrt{E^2 - \lambda e^{2k}}}, \tag{12}$$

$$z = \varepsilon b\int\frac{e^{2k-4u}dr}{\sqrt{E^2 - \lambda e^{2k}}}. \tag{13}$$

To continue our analysis, we need to calculate the metric coefficients. To do this, we solve the Einstein field equations and so, we lead to the following equations

$$k'' + (u')^2 = 0, \tag{14}$$

$$w'' - w'k' + w(u')^2 = 0, \tag{15}$$

$$w'k' - w(u')^2 = 0, \tag{16}$$

$$w'' + wk'' - 2wu'' - 2w'u' + w(u')^2 = 0, \tag{17}$$

the overhead prime indicate differentiation with respect to r. By comparing the equations (15) and (16), one can find

$$w = c_1 r + c_2, \tag{18}$$

in which c_i are integration constants. By substituting the last relation into field equations, we can conclude

$$k = c_3 \ln(\frac{w}{c_1}) + c_4, \tag{19}$$

and two solutions for u become as follows

$$u_1 = \varepsilon\sqrt{c_3}\ln(w) + c_5 \;\;\&\;\; u_2 = \frac{k}{2}. \tag{20}$$

Next, we will determine the trajectory of particle for two following cases:

Case (1): $u = u_1$

First, without loss of generality, we restrict our analysis to the special case $c_3 = 1$. In this case, by considering the equations (11-13) and some tedious calculations, the trajectory of particle is obtained as follows

$$w = \frac{\varepsilon\sqrt{L}}{E}\sqrt{t^2 + \left(\frac{\mu E}{L}\right)^2} = \frac{\varepsilon\sqrt{L}}{a}\sqrt{c_1^4 e^{-4c_4}\phi^2 + \left(\frac{\mu a}{L}\right)^2} = -\left\{\frac{L^2 e^{\frac{i\varepsilon b c_1^2}{\mu}z}}{c_1^2} + \frac{c_1^2 e^{-\frac{i\varepsilon b c_1^2}{\mu}z}}{4\mu^2}\right\}^{-1}, \tag{21}$$

where $L = c_1^2 E^2 - a^2 e^{2c_4} - m^2 e^{2c_4 - 2c_5}$, $\mu = b e^{c_4 - 2c_5}$ and $i = \sqrt{-1}$. Therefore, we can conclude

$$\phi = \frac{a e^{2c_4}}{c_1^2 E}\, t, \tag{22}$$

$$r = \frac{\varepsilon\sqrt{L}}{c_1 E}\sqrt{t^2 + \left(\frac{\mu E}{L}\right)^2} - \frac{c_2}{c_1}, \tag{23}$$

$$z = -\frac{i\varepsilon\mu}{b c_1^2}\ln\left(-\frac{\varepsilon\mu E + \sqrt{\mu^2 E^2(1-L) - t^2 L^3}}{2\mu L\sqrt{t^2 L^3 + \mu^2 E^2 L}}\, c_1^2\right). \tag{24}$$

Also, from the equation (11), the radial velocity of particle has the following expression

$$\frac{dr}{dt} = \frac{\varepsilon\sqrt{E^2 - \lambda e^{2k}}}{E}. \tag{25}$$

The turning points of the trajectory are given by $\frac{dr}{dt} = 0$. As a consequence, the potential curves are

$$E = \lambda e^{k}. \tag{26}$$

We note that if $-\frac{e^{c_4}\sqrt{a^2+m^2e^{-2c_5}}}{c_1}\leq E\leq\frac{e^{c_4}\sqrt{a^2+m^2e^{-2c_5}}}{c_1}$, then there are no bound states and the particle cannot be trapped by the extended object with the static cylindrically symmetric geometry. But if $|E|>\frac{e^{c_4}\sqrt{a^2+m^2e^{-2c_5}}}{c_1}$, then there are two real extremals at

$$r=\pm\frac{\mu}{c_1\sqrt{L}}-\frac{c_2}{c_1}. \tag{27}$$

So, the trajectory of particle is bounded, i.e. particle can be trapped.

Case (2): $u=u_2$

Similarly, we consider $c_3=1$. In this case, after some work, we find that

$$\frac{t}{E}=c_1^2e^{-2c_4}\frac{\phi}{a}=\frac{z}{b}=-\frac{2\varepsilon e^{-\frac{c_4}{2}}}{m^2c_1}\sqrt{(E^2-b^2)c_1^2e^{-c_4}-a^2e^{c_4}-m^2c_1w}. \tag{28}$$

Furthermore, we can easily see that the trapping of particle in this case is not possible.

2.2. Calculation of the gravitoelectromagnetic force

At first, from the equations (11-13), we can deduce

$$^*v^i=\frac{1}{E}\begin{cases}\varepsilon e^{u-k}\sqrt{E^2-\lambda e^{2k}} & i=1,\\ \dfrac{ae^{u+k}}{w^2} & i=2,\\ be^{k-3u} & i=3.\end{cases} \tag{29}$$

With applying this relation, after simplifying, we lead to

$$\frac{m}{\sqrt{1-{}^*v^2}}=Ee^{u-k}. \tag{30}$$

In the next step, all nonzero components of starry Christoffel symbols are calculated as

$$\begin{aligned}
&^*\lambda^1_{11}=k'-u',\\
&^*\lambda^1_{22}=(u'w^2-ww')e^{-2k},\\
&^*\lambda^1_{33}=-u'e^{4u-2k},\\
&^*\lambda^2_{12}=-u'+\frac{w'}{w},\\
&^*\lambda^3_{13}=u',
\end{aligned} \tag{31}$$

in our notation $(r,\phi,z)\equiv(1,2,3)$. At this stage, from the equations (29-31) and using this fact that all components of D_{ij} are zero, we can derive the following expressions

$$\frac{{}^*d^*\mathrm{p}^1}{dt}=\frac{e^{u-k}}{E}\left\{(u'-k')(2E^2e^{2u-2k}-m^2)+\frac{a^2e^{2u}}{w^2}(k'-2u'+\frac{w'}{w})+b^2e^{-2u}k'\right\}, \tag{32}$$

$$^*\mathrm{M}^1=\frac{1}{E^2}\left\{(u'-k')(E^2e^{2u-2k}-m^2)+\frac{a^2e^{2u}}{w^2}(k'-2u'+\frac{w'}{w})+b^2e^{-2u}k'\right\}, \tag{33}$$

$$\frac{{}^*d^*\mathrm{p}^2}{dt}={}^*\mathrm{M}^2=\frac{2a\varepsilon(wu'-w')e^{2u}\sqrt{E^2-\lambda e^{2k}}}{w^3E^2}, \tag{34}$$

$$\frac{{}^*d^*\mathrm{p}^3}{dt}={}^*\mathrm{M}^3=-\frac{2b\varepsilon u'e^{-2u}\sqrt{E^2-\lambda e^{2k}}}{E^2}. \tag{35}$$

Further, we can verify that all components of gravitoelectromagnetic fields are vanish except

$$^*\mathrm{E}_1=u'-k'. \tag{36}$$

By considering the equations (30-36) and some calculations, we finally achieve to

$$^*\mathbf{F}=0. \tag{37}$$

3. Conclusion

The motion of relativistic test particles in the static cylindrically symmetric metric have been investigated. We proved that the particles can be trapped by this gravitational field. Also, it was shown that the gravitoelectromagnetic force acting on particles in this spacetime vanished.

ფიზიკა

რელატივისტური საცდელი ნაწილაკის კლასიკური მოძრაობა ცილინდრულად სიმეტრიულ სტატიკურ მეტრიკაში

მ. იავარი

ახადის ისლამური უნივერსიტეტი, ქაშანის განყოფილება, ქაშანი, ირანი

(წარმოდგენილია აკადემიის წევრის ა. ხელაშვილის მიერ)

ნაშრომში ბოჭკოვანი ფორმალიზმით განხილულია გრავიტოელექტრული და გრავიტომაგნიტური ველები. რელატივისტური საცდელი ნაწილაკის მოძრაობა ცილინდრულად სიმეტრიულ სტატიკურ მეტრიკაში შესწავლილია ჰამილტონ-იაკობის მეთოდით. ბოჭკოვანი ფორმალიზმით გამოთვლილია აგრეთვე გრავიტოელექტრომაგნიტური ძალა ამ სახის მეტრიკულ სივრცე-დროში.

REFERENCES:

1. *C. W. Misner, K. S. Thorne, J. A. Wheeler* (1973), Gravitation. W. H. Freeman and Company, San Francisco,
2. *L. D. Landau and E. M. Lifshitz* (1975), The Classical Theory of Fields, 4th edn., Pergamon Press, Oxford,
3. *L. D. Landau and E. M. Lifshitz* (1941), Teoriia Polia, Moscow (in Russian).
4. *A. Lichnerowicz* (1944), J. Math. Pure Appl., **23:** 37-63.
5. *R. Jantzen, P. Carini* (1991), Understanding Spacetime Splittings and their Relationships in Classical Mechanics and Relativity. In: Relationship and Consistency (G. Ferrarese, Ed.). Bibliopolis, Naples: 185-241.
6. *S. Boersma, T. Dray* (1995), Gen. Relativ. Grav., **27:** 319.
7. *J. Katz, D. Lynden-Bell, J. Bicak* (2006), Class. Quantum Grav. **23:** 7111-7128.
8. *M. Nouri-Zonoz, A. R. Tavanfar* (2003), J. High Energy Phys. **02:** 059.
9. *A. Zel'manov* (1956), Soviet. Phys. Doklady, **1:** 227; Chronometric Invariants, American Research Press, New Mexico, 2006.
10. *B. Mashhoon* (2008), Gravitoelectromagnetism: A Brief Review. arXiv: gr-qc/0311030v2.
11. *R. T. Jantzen, P. Carini, and D. Bini* (1992), Ann. Phys., **215:** 1-50.
12. *H. Stephani, D. Kramer, M. A. H. MacCallum, et al.* (2003), Exact Solutions of Einstein's Field Equations, Cambridge University Press, Cambridge.
13. *S. Chakraborty, L. Biswas* (1996), Class. Quantum Grav., **13:** 2153.
14. *S. Chakraborty* (1996), Gen. Relativ. Grav., **28:** 1115.
15. *J. Gamboa, A. J. Segui-Santonja* (1992), Class. Quantum Grav., **9:** L111.

Received January, 2014